SIZE DIRECTION GAMES OVER THE REAL LINE. I

BY

ANDRZEJ EHRENFEUCHT AND GADI MORAN

ABSTRACT

Two related infinite two person games of perfect information are defined and studied.

0. Introduction

The purpose of this article is to introduce and study certain types of two person games of perfect information over the real line (see [3]). Its second part [2] treats continuity restrictions on the class of strategies of one of the players, and their influence on the collection of winning sets of each of the players. References to related work are given in [1].

Let X be a set of real numbers. We associate with X two games $\Gamma^D(X)$ and $\Gamma^S(X)$ which are defined as follows: Two players S (Size) and D (Direction) construct a sequence of real numbers $\langle s_n : n < \omega \rangle$ in the following way: S chooses s_0. Once s_n is determined, $s_{n+1} = s_n + \varepsilon_n x_n$, where x_n is a positive real number chosen by S, and $\varepsilon_n \in \{-1, 1\}$ is chosen by D. In the game $\Gamma^D(X)$ D chooses ε_n first and then S chooses x_n, whereas in $\Gamma^s(X)$ roles are interchanged: S chooses x_n first and then D chooses ε_n. (Notice that in both games S chooses s_0 first). S wins if $\langle s_n : n < \omega \rangle$ is a convergent sequence, and its limit belongs to X. D wins otherwise.

In §2 we prove two theorems. The first solves Γ^D.

THEOREM 1. *$\Gamma^D(X)$ is a win for D if X is discrete, and is a win for S otherwise.*

Thus, $\Gamma^D(X)$ is a determined game for any X. This should be contrasted with the situation of similar games, which can be nondetermined (see [1]) If, however, S is restricted to use continuous strategies (see [2]) the game may be non-determined. It turns out that under such restrictions, the game is a win for S iff X includes a perfect subset, and a win for D iff X is at most denumerable.

Received Sept. 15, 1971 and in revised form August 30, 1972

Our next theorem shows that D can do realy more in Γ^S than in Γ^D.

THEOREM 2. *$\Gamma^S(X)$ is a win for D if X is at most denumerable.*

It is shown in [2] that $\Gamma^S(X)$ is a win for S if X includes a perfect subset, and that if $\Gamma^S(X)$ is a win for D then X is at most denumerable. By Corollary 2 below, if $\Gamma^S(X)$ is a win for S then X includes a perfect subset. Thus, $\Gamma^S(X)$ is a win for S iff X includes a perfect subset, and it is a win for D iff X is at most denumerable.

A strategy for S in $\Gamma^D(X)$ can be coded in a natural way by a real valued function f over the set 2^* of all finite sequences of zeros and ones, which satisfies

$$f(\xi \cdot \langle 0 \rangle) < f(\xi) < f(\xi \cdot \langle 1 \rangle) \tag{1}$$

for every ξ in 2^* (where $\cdot$ denotes concatenation of sequences). Similarly, a strategy of S in $\Gamma^S(X)$ can be coded by such a function f which satisfies (1) and

$$f(\xi) - f(\xi \cdot \langle 0 \rangle) = f(\xi \cdot \langle 1 \rangle) - f(\xi) \tag{2}$$

for every ξ in 2^*.

This observation leads to the following two corollaries of Theorems 1 and 2. Let f be a real valued function over 2^* such that:

(3) for each infinite sequence of zeros and ones $\langle \varepsilon_n : n < \omega \rangle$, the sequence $\langle f(\langle \varepsilon_0, \cdots, \varepsilon_n \rangle) : n < \omega \rangle$ is a convergent sequence.

Denote by $L(f)$ the set of all limits of such sequences. It can be shown that $L(f)$ is always an analytic set.

COROLLARY 1. *If f satisfies* (1) *and* (3), *then $L(f)$ cannot be finite, but it can be countable.*

COROLLARY 2. *If f satisfies* (1), (2) *and* (3), *then $L(f)$ includes a nonempty perfect subset.*

Thanks are due to Jan Mycielski for his help in the preparation of this paper.

1. Notation

ω denotes the set of non negative integers and R the set of real numbers. R^+ is the set of positive real numbers. We refer to a sequence $\langle s_n : n < \omega \rangle$ constructed in the game $\Gamma^D(X)$ or $\Gamma^S(X)$ as *a play* of this game, and to its limit (when it exists) as *the outcome* of this play.

We call a strategy of a player in one of these games *a positional strategy* if it depends only on the position attained in the step in question (but not on "the history"), and perhaps also on the move taken by the opponent in the present step (in the case that our player makes the second choice in the steps of the games). Thus, for example, a positional strategy for D in Γ^D is a function from R to $\{-1,1\}$, whereas a positional strategy for D in Γ^S is a function from $R \times R^+$ into $\{-1,1\}$, and a positional strategy for S in Γ^D is a function from $R \times \{-1,1\}$ into R^+.

Let z, s be two real numbers. We say that $\varepsilon \in \{-1,1\}$ is a *recoil from* z *at* s if $(z-s)\cdot\varepsilon \leqq 0$. The motivation is clear: if $s = s_n$ is the n'th position in a play, and $s_{n+1} = s_n + \varepsilon_n x_n$ is the $(n+1)$-th position, then s_{n+1} is farther away from z than s_n if ε_n is a recoil from z at s_n.

2. Proofs of Theorems 1 and 2

We shall explicity define a positional winning strategy for the winner, for each of the three assertions "if X is $\cdots$ then $\cdots$ has a winning strategy in $\cdots$".

PROOF OF THEOREM 1.

1) *If X is discrete, then D has a positional winning strategy in* $\Gamma^D(X)$:

PROOF. $X \subseteq R$ is discrete iff there is a family $\Omega = \{g_z : z \in X\}$ of open intervals such that:

(i) $z \in g_z$

(ii) if $z, z' \in X$ and $z \neq z'$ then $g_z \cap g_{z'} = \emptyset$.

Fix such an Ω for X. The positional winning strategy is defined as follows: If $s_n \notin \cup\,\Omega$, then $\varepsilon_n = 1$. Otherwise, let $z \in X$ be the unique element for which $s_n \in g_z$; then ε_n is a recoil from z at s_n.

Assume now, that $s = \langle s_n : n < \omega \rangle$ is a play were D uses the described strategy. If s is divergent, D won. Assume that s is a convergent sequence, and $s = \lim s_n$. We shall show that $s \notin X$ and hence, again, D won. Indeed, if $s \in X$, then for some $N \in \omega$, $s_n \in g_s$ for all $n \geqq N$. Let N_0 be the first such N. Then by the choice of N_0 and the definition of our strategy, $s_{N_0} \in g_s$, $s_{N_0+1} \in g_s - \{s\}$, and for $n > N_0$, ε_n is a recoil from s at s_n. So we have for $n > N_0$:

$$\left| s_n - s \right| \geqq \left| s_{N_0+1} - s \right| > 0$$

which contradicts "$s = \lim s_n$".

2) *If X is not discrete, then S has a positional winning strategy in* $\Gamma^D(X)$.

PROOF. Assume that X is not discrete. Hence there exists a monotonous sequence $\langle z_k : k < \omega \rangle$ of elements of X such that $z = \lim z_k$ also belongs to X. Assume further, that $\langle z_k : k < \omega \rangle$ is a monotonous increasing sequence. The other case is treated similarly. Put $y_k = \frac{1}{2}(z_k + z_{k+1})$.

The winning strategy for S is described as follows:

$$s_0 = y_0 .$$

Assume, by induction, that for s_n consistent with our strategy, there is a (unique) z_k such that $z_k < s_n < z_{k+1}$. Call z_k *the neighbor* of s_n. Let $\varepsilon_n \in \{-1, 1\}$ be given. If $\varepsilon_n = -1$, x_n is chosen so that $s_{n+1} = \frac{1}{2}(z_k + s_n)$. If $\varepsilon_n = 1$, x_n is chosen so that $s_{n+1} = y_{k+1}$, i.e.:

$$x_n = \begin{cases} (s_n - z_k) & \varepsilon_n = -1, \\ y_{k+1} - s_n & \varepsilon_n = 1. \end{cases}$$

We see that our induction hypothesis is carried on. Moreover, if $\varepsilon_n = -1$, then s_{n+1} and s_n have the same neighbor, namely z_k, and if $\varepsilon_n = 1$, then z_{k+1} is the neighbor of s_{n+1}.

To see that this is a winning strategy for S, let any sequence $\langle \varepsilon_n : n < \omega \rangle \in \{-1,1\}^\omega$ be given, and we show that the play where D chooses this sequence as his sequence of moves is a convergent sequence with limit in X.

We distinguish two cases:

(a) For some N_0, $\varepsilon_n = -1$ for $n \geqq N_0$. It is clear that in this case the play is a sequence converging to the commmon neighbour of the s_n's with $n \geqq N_0$, which is a member of X.

(b) There is an infinite increasing sequence of natural numbers $\langle n_k : k < \omega \rangle$ such that for all $m < \omega$, $\varepsilon_m = 1$ iff $m = n_k$ for some $k < \omega$.

One proves by induction that for $n_k < m \leqq n_{k+1}$, z_{k+1} is the neighbor of s_m. This implies that $\lim s_n = \lim z_k = z \in X$. □

Proof of Theorem 2

3) *If X is at most denumerable, then D has a winning strategy in $\Gamma^S(X)$.*

Proof. We may assume that X is not finite. Let $\{z_n : n < \omega\}$ be an enumeration of X. Construct by induction a sequence of open intervals $\langle g_n : n < \omega \rangle$ such that:

(i) $z_n \in g_n \subset (z_n - 1/n, z_n + 1/n)$.

(ii) the endopoints of g_n are not in X.

(iii) if $m < n$ then $g_n \subset g_m$ or else $g_n \cap g_m = \emptyset$.

(iv) if $m < n$ then $z_m \notin g_n$.

Observe that (ii) for all $m < n$ implies that the distance from z_n to the endpoints of g_m is a positive number, and hence the construction can be carried on so that (iii) holds.

For $s \in R$ and $x \in R^+$, define $\varepsilon(s,x)$ as follows:

If for all $n < \omega$ $(s - x, s + x) \nsubseteq g_n$ then $\varepsilon(s,x) = 1$.

Otherwise, let n be maximal such that $(s - x, s + x) \subseteq g_n$; then $\varepsilon(s,x)$ is a recoil from z_n at s.

We shall show now that $\varepsilon(s,x)$ is a positional winning strategy for D. Let $\langle s_n : n < \omega \rangle$ be a play where $\varepsilon_n = \varepsilon(s_n, x_n)$ for all $n < \omega$, and assume that $\lim s_, = z_p$ for some $p < \omega$. Put

$$\Omega' = \{g_m : m > p \text{ and } g_m \subset g_p\}$$

and let $\Omega \subset \Omega'$ be the family of the maximal intervals of Ω' with respect to inclusion. Thus, Ω is a family of pairwise disjoint subintervals of g_p, and z_p does not belong to the closure of any member of Ω.

Let $N \in \omega$ satisfy: for all $n \geqq N$ $(s_n - x_n, s_n + x_n) \subset g_p$ (where $x_n = |s_{n+1} - s_n|$ as usual). Using the definitions of $\varepsilon(s,x)$ and of Ω it is easy to verify the following facts:

(1) If for some $N_1 \geqq N$ $(s_n - x_n, s_n + x_n) \nsubseteq g$ for all $n \geqq N_1$ and $g \in \Omega$, then for all $n \geqq N_1$

$$|s_{n+1} - z_p| = |s_n - z_p| + x_n > |s_n - z_p|.$$

(2) If for some $n \geqq N$, $g \in \Omega$, $s_n \in g$ and $s_{n+1} \notin g$, then for any $k > 0$

$$|s_{n+k} - z \cdot| \geqq \sup\{|t - z \cdot| : t \in g\} > 0.$$

From (1) it follows that for some $n \geqq N_1$, s_n should fall in a member g of Ω, and by (2) s_{n+k} should belong to g for $k \in \omega$. But then $\lim s_n$ is in the closure of g, so it cannot be equal to z_p. □

References

1. G. Moran, *Existence of nondetermined sets for some two person games over reals*, Israel J. Math. 9 (1971), 316–329.

2. G. Moran, *Size Direction games over the real line. II*, Israel J. Math. to appear,

3. J. Mycielski, *Continuous games with perfect information*, Advances in Game Theory, Princeton (1964), 103–112.

DEPARTMENT OF COMPUTER SCIENCES
UNIVERSITY OF COLORADO
BOULDER, COLORADO, U.S.A.

AND

DEPARTMENT OF MATHEMATICS
HEBREW UNIVERSITY
JERUSALEM, ISRAEL